Conservation law for multimoded nonlinear optical waveguide interactions and its physical interpretation

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The swapping of power between the modes of cw waves in lossless nonlinear optical waveguides always admit two conserved quantities, the total power and one other, which is sometimes identified as a Hamiltonian. We show that a general formulation of this Hamiltonian is in fact the weak guidance limit for cw waves of a more general conservation law. We make the link between this more general conservation law and the conservation of ''wave'' momentum, where wave momentum is a combination of both real momentum and so-called pseudomomentum. This allows us to interpret the conserved Hamiltonian in physical terms. [S1063-651X(99)03405-4]

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I. INTRODUCTION

Few mode nonlinear optical waveguide devices have received considerable attention in recent years because of their potential application as very high speed switches in communications networks (see Refs. [1–17], for example). Of considerable importance in the theoretical analysis of such devices are the conserved quantities of the system, as they allow the dimension of the mathematical problem to be reduced, so that a few, or even a single differential equation or integration leads to a closed-form solution for the dynamical path of the system in its phase space. For cw interactions in lossless devices, conserved power is always one of the constants of motion for example, and its physical interpretation in terms of the conservation of energy is obvious. In multiple frequency interactions, the Manley-Rowe relations also provide constants of motion, whose physical interpretation is in terms of conserved photon numbers.

Apart from conserved total power, and the Manley-Rowe relations when they are applicable, there is always another conserved quantity for cw multimode interactions in a lossless waveguide [18]. This constant either ''just comes out of the equations'' [3–10], or is formulated as a ''Hamiltonian'' H [11–15] (though not to be confused with the classical mechanics Hamiltonian of total energy). Despite its widespread use however, the precise physical interpretation of this constant has never been fully investigated, though it has been suggested to be related to stored energy [17,19], momentum [16,20,21], and momentum flow [18] by various authors. A primary purpose of this paper therefore, is to answer definitively the question of the proper physical interpretation of H. In doing so, we will also show that this constant follows from a more general conservation law derivable directly from Maxwell's equations without recourse to coupled mode formalism. For simplicity, we shall limit our analysis to the case of an ideal Kerr law medium when third harmonic generation can be neglected, though the final interpretation is believed to be completely general.

For the case under discussion, the Hamiltonian constant of motion can be written in the general form [18]

$$H=\sum_{k=1}^{N}\left(\frac{\beta_k P_k}{\omega_0}\right)+\frac{1}{4}\int_{A_\infty}\langle \mathbf{P}_t^{\mathrm{NL}}\cdot\mathbf{E}_t^{*}+\mathbf{P}_t^{\mathrm{NL}*}\cdot\mathbf{E}_t\rangle dA \qquad (1)$$

for waveguides translationally invariant in the z direction. In this equation, ω_0 is the angular frequency of the fields, β_k is the linear propagation constant of the kth mode, $P_k(z)$ is the power in the kth mode at waveguide position z, the superscript asterisk denotes complex conjugate, $\mathbf{P}_t^{\mathrm{NL}}$ is the induced nonlinear polarization due to the transverse part $\mathbf{E}_t$ of the total electric field, and the angled brackets indicate a suitable time average.

In the next section, we show directly from Maxwell's equations that H is the weak guidance limit for cw waves of a more general conservation law. In Sec. III, we then show that this more general conservation law is in fact the z component of the conservation law for ''wave momentum'' in a rigid dielectric. This then allows us to answer the question of the proper physical interpretation of H.

II. GENERAL CONSERVATION LAW

A. General theory

For our purposes, we need Maxwell's equations for an electric and magnetic field $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ for charge free nonmagnetic stationary media (e.g., optical fibers/dielectric waveguides). These are given by

$$\nabla\times\boldsymbol{\mathcal{E}}=-\frac{\partial\boldsymbol{\mathcal{B}}}{\partial t}, \qquad (2a)$$

$$\nabla\times\boldsymbol{\mathcal{H}}=\frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}, \qquad (2b)$$

$$\nabla\cdot\boldsymbol{\mathcal{D}}=0, \qquad (2c)$$

$$\nabla\cdot\boldsymbol{\mathcal{B}}=0, \qquad (2d)$$

where

$$\boldsymbol{\mathcal{B}}=\mu_0\boldsymbol{\mathcal{H}}, \qquad (3a)$$

$$\boldsymbol{\mathcal{D}} = \epsilon_0 \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}}, \tag{3b}$$

and $\boldsymbol{\mathcal{D}}$ is the total displacement, $\boldsymbol{\mathcal{B}}$ is the total magnetic flux, $\boldsymbol{\mathcal{P}}$ is the total polarization, μ_0 is the permeability of free space and ϵ_0 is the permittivity of free space. (Note that we use script letters to denote the *real* vector field variables and later on we will introduce normal text letters for the field variables when we rewrite them in complex form. This differentiation between the real and complex forms of the field variables is necessary because we will be dealing with nonlinear dielectrics.)

We use as our ansatz the knowledge that the differential form of conservation laws for continuously distributed (in the continuum limit) vector quantities, such as momentum, have the general form

$$\frac{\partial}{\partial t}(\text{vector current density}) + \boldsymbol{\nabla} \cdot (\text{tensor}) = 0 \tag{4}$$

where for momentum, the tensor is a momentum flux density plus stress tensor.

Considering Maxwell's equations, we find that the obvious candidates for a $\partial(\text{something})/\partial t$ for our desired conservation law include $\partial(\epsilon_0 \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}})/\partial t$, $\partial(\boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{B}})/\partial t$ and $\partial(\boldsymbol{\mathcal{D}} \times \boldsymbol{\mathcal{B}})/\partial t$. Further investigations of each of these candidates reveals that it is the $\partial(\boldsymbol{\mathcal{D}} \times \boldsymbol{\mathcal{B}})/\partial t$ possibility which leads to the desired conservation law as we will now show.

Using the product rule to expand $\partial(\boldsymbol{\mathcal{D}} \times \boldsymbol{\mathcal{B}})/\partial t$ and then substituting from the Maxwell equations (2a) and (2b) leads to

$$\frac{\partial}{\partial t}(\boldsymbol{\mathcal{D}} \times \boldsymbol{\mathcal{B}}) = (\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}}) \times \boldsymbol{\mathcal{B}} + (\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}}) \times \boldsymbol{\mathcal{D}}. \tag{5}$$

To get Eq. (5) looking more similar to Eq. (4), we use the vector/tensor identity Eq. (A5) from Appendix A to rewrite the right-hand side of Eq. (5) in terms of divergences and other quantities. Additionally using Eq. (3a) and the Maxwell Eqs. (2c) and (2d), results finally in the identity:

$$\frac{\partial}{\partial t}(\boldsymbol{\mathcal{D}} \times \boldsymbol{\mathcal{B}}) + \boldsymbol{\nabla} \cdot \mathbf{T} = \frac{1}{2} \boldsymbol{\mathcal{E}} \cdot (\boldsymbol{\nabla} \boldsymbol{\mathcal{D}}) - \frac{1}{2} \boldsymbol{\mathcal{D}} \cdot (\boldsymbol{\nabla} \boldsymbol{\mathcal{E}}) \tag{6a}$$

$$\equiv \frac{1}{2} \boldsymbol{\mathcal{E}} \cdot (\nabla \boldsymbol{\mathcal{P}}) - \frac{1}{2} \boldsymbol{\mathcal{P}} \cdot (\nabla \boldsymbol{\mathcal{E}}), \tag{6b}$$

where

$$\mathbf{T} = \frac{1}{2} \mathbf{I}(\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{B}}) - \boldsymbol{\mathcal{E}} \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{H}} \boldsymbol{\mathcal{B}} \tag{7}$$

is the negative of the Maxwell stress tensor [22] and the spatial part of the Minkowski energy-momentum tensor [23].

For the standard case of a homogeneous, dispersion-free linear medium, $\boldsymbol{\mathcal{D}} = \epsilon^L \boldsymbol{\mathcal{E}}$ (where ϵ^L is the constant linear dielectric permittivity of the medium), and the right-hand side of Eq. (6) is identically zero, thus giving the "momentum" conservation law of Minkowski [23]. However, in general, Eq. (6) does not fit the form of Eq. (4), and so is not a conservation law but a "balance" law.

B. An important special case

In a nonhomogeneous dispersion-free linear medium (for which $\boldsymbol{\mathcal{D}} = \epsilon^L \boldsymbol{\mathcal{E}}$ but now ϵ^L is a function of position), the right-hand side of Eq. (6) reduces to $(1/2)\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}(\nabla \epsilon^L)$. It is the presence of this term in $\boldsymbol{\nabla} \epsilon^L$ which prevents Eq. (6) from being a conservation law in general. (We note in anticipation at this point, that the requirement that the medium be homogeneous for a "balance" law to become a conservation law is a hallmark of something which has been dubbed pseudomomentum or quasimomentum. We postpone a discussion of pseudomomentum to Sec. IIIA.)

A conservation law can still be obtained from Eq. (6) for a special, though practically important, class of nonhomogeneous media, however—namely, waveguides which are uniform in the z direction. For such waveguides, the z component of Eq. (6) integrated over the infinite cross section of the waveguide will yield a conservation law, even when the effects of dispersion and nonlinearity are accounted for, as we will now show.

First we break $\boldsymbol{\mathcal{D}}$ into linear and nonlinear parts, i.e., $\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}^L + \boldsymbol{\mathcal{P}}^{\mathrm{NL}}$, where $\boldsymbol{\mathcal{P}}^{\mathrm{NL}}$ is the nonlinear part of the polarization density vector. Using Eq. (7), this allows $\boldsymbol{\nabla} \cdot \mathbf{T}$ in Eq. (6) to be split into linear and nonlinear parts as follows:

$$\boldsymbol{\nabla} \cdot \mathbf{T} \equiv \boldsymbol{\nabla} \cdot \mathbf{T}^L + \frac{1}{2} \boldsymbol{\mathcal{P}}^{\mathrm{NL}} \cdot (\nabla \boldsymbol{\mathcal{E}}) + \frac{1}{2} \boldsymbol{\mathcal{E}} \cdot (\boldsymbol{\nabla} \boldsymbol{\mathcal{P}}^{\mathrm{NL}}) - \boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{E}} \boldsymbol{\mathcal{P}}^{\mathrm{NL}}), \tag{8}$$

where we have also used Eq. (A3b). Substituting this result into Eq. (6a), taking the dot product with $\hat{\mathbf{z}}$ and then integrating the result over the infinite cross section A_∞ of the waveguide, we get

$$\int_{A_\infty} dA \left[\frac{\partial}{\partial t}(\boldsymbol{\mathcal{D}}_t \times \boldsymbol{\mathcal{B}}_t) \cdot \hat{\mathbf{z}} + \frac{\partial}{\partial z}(T^L_{zz} - \mathcal{E}_z \mathcal{P}^{\mathrm{NL}}_z) + \boldsymbol{\mathcal{P}}^{\mathrm{NL}} \cdot \frac{\partial \boldsymbol{\mathcal{E}}}{\partial z} + \frac{1}{2} \boldsymbol{\mathcal{D}}^L \cdot \frac{\partial \boldsymbol{\mathcal{E}}}{\partial z} - \frac{1}{2} \boldsymbol{\mathcal{E}} \cdot \frac{\partial \boldsymbol{\mathcal{D}}^L}{\partial z} \right] = 0, \tag{9}$$

where the subscript t represents the transverse part of the field, $\hat{\mathbf{z}}$ is a unit vector in the z direction, and we have used the results $[\mathbf{a} \cdot (\nabla \mathbf{b})] \cdot \hat{\mathbf{z}} \equiv \mathbf{a} \cdot \partial \mathbf{b}/\partial z$, and for bound fields (i.e., fields that go to zero at infinity) the two-dimensional (2D) divergence theorem gives that

$$\int_{A_\infty} dA\, \hat{\mathbf{z}} \cdot (\nabla \cdot \mathbf{T}) = \int_{A_\infty} dA \frac{\partial T_{zz}}{\partial z}, \tag{10}$$

where T_{zz} is the zz component of $\mathbf{T}$.

In order to determine the effects of dispersion (we take account of dispersion in $\boldsymbol{\mathcal{D}}^L$, but neglect dispersion in $\boldsymbol{\mathcal{P}}^{\mathrm{NL}}$ [24]), we next assume that the time dependence in $\boldsymbol{\mathcal{E}}$ can be separated into a rapidly varying part with angular frequency ω_0 and a slowly varying envelope function $\mathbf{E}$ so that

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) e^{-i\omega_0 t} + \mathbf{E}^*(\mathbf{r},t) e^{i\omega_0 t}, \tag{11}$$

where

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi} \int_0^\infty d\omega\, \boldsymbol{\mathcal{E}}_\omega e^{-i(\omega-\omega_0)t} \tag{12}$$

and similar expressions also apply to all the other field quantities.

Replacing all fields in Eq. (9) by their expanded forms [i.e., by Eq. (11) or its equivalent] and taking a suitable time average [25], we end up with the following conservation law for a cubic nonlinear medium:

$$\frac{\partial \mathcal{G}}{\partial t}+\frac{\partial \mathcal{T}}{\partial z}=0, \tag{13}$$

where the scalars

$$\mathcal{T}=\int_{A_\infty} dA\bigg[\langle T_{zz}^L\rangle - E_z^* P_z^{\rm NL} - E_z P_z^{NL*} + \frac{1}{4}(\mathbf{P}^{\rm NL}\cdot\mathbf{E}^* + \mathbf{P}^{\rm NL*}\cdot\mathbf{E})\bigg], \tag{14a}$$

$$\mathcal{G}=\int_{A_\infty} dA\bigg[(\mathbf{D}_t^*\times\mathbf{B}_t+\mathbf{D}_t\times\mathbf{B}_t^*)\cdot\hat{\mathbf{z}} + \frac{i}{2}\frac{\partial\epsilon^L}{\partial\omega}\bigg(\mathbf{E}\cdot\frac{\partial\mathbf{E}^*}{\partial z}-\mathbf{E}^*\cdot\frac{\partial\mathbf{E}}{\partial z}\bigg)\bigg], \tag{14b}$$

and

$$\langle T_{zz}^L\rangle = \mathrm{Re}\{\mathbf{E}_t\cdot\mathbf{D}_t^{L*}+\mathbf{H}_t\cdot\mathbf{B}_t^* - E_z D_z^{L*} - H_z B_z^*\}, \tag{14c}$$

where Re denotes the real part and we have also used Eqs. (B2) and (C4) from the appendixes to rewrite the last three terms on the left-hand side of Eq. (9).

Note that in the linear limit for a single mode propagating down a uniform dielectric waveguide, Eqs. (13) and (14) reduce to the results found by Haus and Kogelnik [26]. They also reduce to the results found by Nelson [27] and Loudon *et al.* [28] for an electromagnetic wave propagating in a linear homogeneous medium. To compare with analyses that have ignored dispersion, just set $\partial\epsilon^L/\partial\omega\equiv 0$ in Eq. (14b).

C. CW limit

For a cw wave, the field envelope quantities are constant in time, and so the conservation law given by Eq. (13) reduces to the "constant of motion"

$$\mathcal{T}=\text{const}, \tag{15}$$

where $\mathcal{T}$ is still given by Eq. (14a). Note that this result is still at this point completely general and free from approximations apart from the assumptions that the waveguide be lossless, translationally invariant, and composed of an ideal Kerr law material. We have also assumed that electrostriction, magnetostriction, third harmonic generation, and radiation fields may be neglected.

D. Constant of motion in terms of the linear modes of the waveguide

If we now consider $\boldsymbol{\mathcal{P}}^{\rm NL}$ to be a small perturbation to the linear waveguide, then we can expand the total fields [given by expressions such as Eq. (11), but where now the field envelope quantities are constants in time] in terms of the modes of the unperturbed (linear) waveguide as follows [29,30]:

$$\mathbf{E}_t=\sum_j a_j\mathbf{e}_{tj}(x,y)e^{i\beta_j z}, \tag{16a}$$

$$\mathbf{H}_t=\sum_j a_j\mathbf{h}_{tj}(x,y)e^{i\beta_j z}, \tag{16b}$$

$$E_z=\sum_j \frac{n^2}{\tilde{n}_j^2}a_j e_{zj}(x,y)e^{i\beta_j z}, \tag{16c}$$

$$H_z=\sum_j a_j h_{zj}(x,y)e^{i\beta_j z}, \tag{16d}$$

where a_j is the amplitude of the jth modal field which has a propagation constant β_j, and we have split the linear (unperturbed) waveguide modal fields $\mathbf{e}_j=\mathbf{e}_{tj}+e_{zj}\hat{\mathbf{z}}$ and $\mathbf{h}_j=\mathbf{h}_{tj}+h_{zj}\hat{\mathbf{z}}$, into their transverse (subscript t) and longitudinal (subscript z) components. Note that since we have expanded the transverse parts of the fields in terms of the unperturbed modal fields (since those fields form a complete set for the total transverse field) in order for the longitudinal component of $\mathbf{E}$ to be consistent with both this expansion and Maxwell's equations, it is not given by an expansion in terms of the unperturbed longitudinal modal fields, but rather by the modified form shown in Eq. (16c) [29]. Thus in Eq. (16c), $n(x,y)$ is the linear (unperturbed) refractive index of the waveguide whilst $\tilde{n}_j(x,y,z)$ is the perturbed refractive index experienced by mode j, i.e., $\tilde{n}_j$ depends on the power in each of the modes through the nonlinearity.

If we write $\tilde{n}_j^2=n^2+\delta n_j^2$, where $\delta n_j^2/n^2\ll 1$, then the perturbed waveguide modal displacement field $\tilde{d}_{zj}\equiv\epsilon_0 n^2(n^2/\tilde{n}_j^2)e_{zj}\simeq d_{zj}-\epsilon_0\delta n_j^2 e_{zj}$, where $d_{zj}=\epsilon_0 n^2 e_{zj}$ is the unperturbed waveguide modal displacement field. Substituting the modal expansions given in Eqs. (16) into the $\langle T_{zz}^L\rangle$ part of Eq. (14a) for $\mathcal{T}$ and using the approximation introduced above, we find that

$$\mathcal{T}\simeq\int_{A_\infty} dA\bigg[\sum_j |a_j|^2\,\mathrm{Re}\{\mathbf{e}_{tj}\cdot\mathbf{d}_{tj}^*+\mathbf{h}_{tj}\cdot\mathbf{b}_{tj}^*-e_{zj}d_{zj}^*-h_{zj}b_{zj}^*\} - E_z^*P_z^{\rm NL}-E_zP_z^{\rm NL*}+\frac{1}{4}(\mathbf{P}^{\rm NL}\cdot\mathbf{E}^*+\mathbf{P}^{\rm NL*}\cdot\mathbf{E}) + (\text{terms in } \delta n^2 \text{ and the longitudinal fields } e_{zj} \text{ and } d_{zk})\bigg]. \tag{17}$$

Rather amazingly, the expression in curly brackets in Eq. (17) is simply a sum over modes because the mixed mode terms that might have been expected to appear in this expression can be shown to evaluate to zero when integrated over the infinite cross section of the waveguide [31].

We now recall that several authors have shown that [30,32,33]

$$\int_{A_\infty} dA |a_j|^2 \,\mathrm{Re}\{\mathbf{e}_{tj}\cdot\mathbf{d}_{tj}^* + \mathbf{h}_{tj}\cdot\mathbf{b}_{tj}^* - e_{zj}d_{zj}^* - h_{zj}b_{zj}^*\}$$
$$= \beta_j P_j/\omega_0,$$

where P_j is the power carried by the jth mode, and so substituting this result into Eq. (17) and *then* taking the weak guidance limit by neglecting all the longitudinal fields, we find that $\mathcal{T}$ reduces to the Hamiltonian H of Eq. (1). We thus see that the conserved Hamiltonian H is the weak guidance limit for cw waves of the more general conservation law given by Eq. (13).

E. An important conceptual point

We now turn to an important conceptual point. Although in the end we have neglected the longitudinal field components, *they were absolutely essential* in determining that $\langle T_{zz}^L\rangle$ is the *difference* between the stored energies per unit length in the transverse and longitudinal modal fields and not the sum. Thus $\langle T_{zz}^L\rangle$ cannot be interpreted as a stored energy term as some authors have done [17,19], but rather it is a wave momentum flow term [26]. We now turn to the determination of the physical interpretation of the general conservation law given by Eq. (13), and hence to the proper physical interpretation of the Hamiltonian H.

III. PHYSICAL INTERPRETATION

We have shown to this point that a canonical form of the conserved ''Hamiltonian'' frequently used in the study of few-mode interactions in weakly guiding nonlinear waveguides does in fact follow from a more general conservation law which we have derived from manipulations of Maxwell's equations. These manipulations, however, leave unanswered the question of what precisely is the proper *physical* interpretation of the conserved quantity. It is to this question that we now turn. Although the conservation law given by Eq. (13) was derived assuming a stationary, rigid dielectric, in order to appreciate its physical meaning, we must first consider an elastic (deformable) medium and then take the limit as this elastic medium becomes rigid.

A. Some general concepts

Our system comprises interacting matter and electromagnetic subsystems. Many of the key ideas needed for the understanding of the physical meaning of the conservation law given by Eq. (13), however, are dealt with elegantly and more simply in a paper by Herrmann [34] in a general discussion of the simpler case of an elastic medium alone. In summary, the key results from this discussion of elastodynamics from the Lagrangian perspective are as follows. (i) The Euler-Lagrange equations of motion may be considered to be a balance law for ordinary momentum, (ii) the time derivative of the Lagrangian density leads to a balance law for energy, and (iii) the space derivatives of the Lagrangian density leads to a balance law for so-called pseudomomentum. From these balance laws, it follows that momentum conservation is a consequence of the homogeneity of space, energy conservation results if the Lagrangian density is explicitly independent of time, and pseudomomentum is conserved if the Lagrangian density is explicitly independent of the material coordinates, i.e., if the medium is homogeneous. Extending these ideas to our matter-field system, total matter-plus-electromagnetic field momentum can be expected to be always conserved. Total pseudomomentum, however, will only be conserved if the medium is homogeneous, which, in the case of our z-invariant waveguide, means that only the z component of pseudomomentum will be conserved. Since the conservation law derived above only applies in the z direction, it follows from the above discussion that pseudomomentum can be expected to play a part in its physical interpretation.

Since pseudomomentum is not a generally familiar concept in the optics community, and there is not complete consensus on its definition in the literature, it is worth saying a few more words about it. As its name suggests, pseudomomentum is not a real momentum, though it has the same dimensions and in many ways behaves as real momentum does—for example, obeying conservation laws and Newtonian-like balance laws involving pseudoforces [34,35]. The most familiar example of pseudomomentum is the ''momentum'' $\hbar\mathbf{k}$ carried by a phonon in a crystal [36], thus indicating the general importance of the concept. The reader is warned, however, that there are two alternative definitions of pseudomomentum in the literature. The one adopted in this paper, is that pseudomomentum is a purely material frame (Lagrangian coordinates) quantity as proposed in Refs. [27,34,35,37]. Other authors however, use relative displacement as their matter field variable [38–41]. This mixes together spatial frame and material frame coordinates, and so mixes together momentum and pseudomomentum as defined by the convention we follow.

B. Wave momentum

The ''momentum'' associated with an electromagnetic wave propagating in a dielectric medium is a topic which has received considerable attention over the years (see, for example, the review papers by Robinson [42] and Brevik [23], the book by Penfield and Haus [43], and the following recent articles [27,28,37–39,44]) though the paper by Nelson [27] contains all the results we shall need. (The interested reader is also directed to Ref. [28], which extends the work of Nelson [27] to include the effects of loss, but also uses a much simpler model for the material medium to derive the desired results.)

In Ref. [27], Nelson first considers the interaction of an electromagnetic wave with a completely general homogeneous elastic dielectric, and derives from the Lagrangian perspective the conservation laws of momentum and pseudomomentum for such a system. He then notes that for an optical frequency wave in a material medium, there is no deformation of the medium. In this special case, the distinction between spatial and material frames vanishes. Consequently, momentum (an inherently spatial (local or Eulerian coordinates) frame quantity) and pseudomomentum (an inherently material frame quantity) can be added to give a new conserved quantity which he has called ''wave momentum.'' Neglecting a magnetization term, the conservation law for wave momentum is given by [27]

$$\frac{\partial}{\partial t}\left(\boldsymbol{\mathcal{D}}\times\boldsymbol{\mathcal{B}}-\sum_{\nu} m^{\nu}\dot{\mathbf{y}}^{\nu}\cdot(\nabla\mathbf{y}^{\nu})\right)+\nabla\cdot(\mathbf{T}+\boldsymbol{\sigma})=\mathbf{0}, \quad (18)$$

where $\mathbf{T}$ is given by Eq. (7), and the tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma}=\mathbf{I}\left(\sum_{\nu}\frac{m^{\nu}}{2}(\dot{\mathbf{y}}^{\nu})^2-\rho^0\Sigma+\frac{1}{2}\boldsymbol{\mathcal{P}}\cdot\boldsymbol{\mathcal{E}}\right). \quad (19)$$

In these two equations, ρ^0 is the mass density (per undeformed volume) of the crystal, m^{ν} is the mass density (per undeformed volume) associated with the ν-internal coordinate $\mathbf{y}^{T\nu}$, the dot represents a material time derivative, and Σ is the stored energy per unit mass. The $N-1$ ($\nu=1,2,\ldots,N-1$) internal coordinates $\mathbf{y}^{T\nu}$ arise because there are assumed to be N particles per primitive unit cell. These internal coordinates are defined as displacement-invariant coordinates so that no (ordinary) momentum is associated with them. The link between $\mathbf{y}^{T\nu}$ and $\mathbf{y}^{\nu}$ is given by $\mathbf{y}^{T\nu}\equiv\mathbf{Y}^{\nu}+\mathbf{y}^{\nu}$, where $\mathbf{Y}^{\nu}$ is the value of $\mathbf{y}^{T\nu}$ in the natural state.

To complete the analysis, it now only remains for us to show that Eq. (18) is in fact equivalent to Eq. (6b). Comparing the two equations, we thus see that we must show that

$$\frac{\partial}{\partial t}\left(\sum_{\nu} m^{\nu}\dot{\mathbf{y}}^{\nu}\cdot(\nabla\mathbf{y}^{\nu})\right)-\nabla\cdot\boldsymbol{\sigma}\equiv\frac{1}{2}\boldsymbol{\mathcal{E}}\cdot(\nabla\boldsymbol{\mathcal{P}})-\frac{1}{2}\boldsymbol{\mathcal{P}}\cdot(\nabla\boldsymbol{\mathcal{E}}). \quad (20)$$

This can be done by expanding the time and space derivatives of the terms on the left-hand side of this equation and noting that for a stationary homogeneous medium in the absence of deformation [27]: $\nabla\mathbf{y}^{\nu}=\nabla\mathbf{y}^{T\nu}$; $\boldsymbol{\mathcal{P}}=\Sigma_{\nu}q^{\nu}\mathbf{y}^{T\nu}$, where q^{ν} is the charge density associated with the internal coordinate $\mathbf{y}^{T\nu}$, $\rho^0\Sigma$ is a function of $\mathbf{y}^{T\nu}$ only, and the internal motion equation (keeping only terms to dipole order) is given by $m^{\nu}\ddot{\mathbf{y}}^{\nu}=-\partial(\rho^0\Sigma)/\partial\mathbf{y}^{T\nu}+q^{\nu}\boldsymbol{\mathcal{E}}$.

This then proves that Eq. (6b) is the conservation law for wave momentum (a sum of real momentum and pseudomomentum) for a homogeneous nondeforming dielectric. Since uniform dielectric waveguides are only homogeneous in the z direction, only the z component of this conservation law (integrated over the infinite cross section of the waveguide) therefore remains a conservation law for uniform dielectric waveguides, as was shown in deriving Eq. (13) in Sec. IIB.

The proper physical interpretation of the Hamiltonian H that we started the paper with is now clear. It is the conserved *wave* momentum flow or momentum flux (i.e., the *wave* momentum flux density integrated over the infinite cross section of the waveguide) for cw waves in the weak guidance limit. Given this interpretation, it is clear that the first term in H, i.e., $\Sigma_j\beta_jP_j/\omega_0$, is the linear wave momentum flow (as was shown for a single waveguide mode in [26]), and the second part, i.e., $\frac{1}{4}\int_{A_\infty}\langle\mathbf{P}_t^{\mathrm{NL}}\cdot\mathbf{E}_t^*+\mathbf{P}_t^{\mathrm{NL}*}\cdot\mathbf{E}_t\rangle dA$, is the nonlinear perturbation correction to the linear wave momentum flow. It is this nonlinear correction which couples energy between the linear modes of the waveguide.

IV. CONCLUSIONS

In this paper, we have extended the conservation law derived by Haus and Kogelnik [26] for a linear dielectric waveguide to the case of a cubically nonlinear waveguide [Eqs. (13) and (14)]. We have also established that this new conservation law has practical as well as theoretical value by showing that it leads, for cw waves interacting at a single frequency, to a general form of the previously established conserved ''Hamiltonian'' H [Eq. (1)] [12,18], for multiple mode interactions in a weakly guiding, weakly nonlinear waveguide. It has been shown that H does in fact follow from a general conservation law, and this is expected to be of value in determining the physical basis of conserved quantities for waves with time-varying envelopes such as temporal solitons.

More immediately, establishing that H follows from a general conservation law has allowed us to call on the works of Nelson and others [27,28,37] to finally resolve the issue of the proper physical interpretation of H [45]. To wit, H, as defined by Eq. (1), is the conserved wave momentum (ordinary momentum plus pseudomomentum) flow for cw waves propagating in a uniform (z invariant) waveguide in the absence of deformation. (The z invariance is required for the conservation of pseudomomentum in this direction, and the absence of deformation is required so that ordinary momentum and pseudomomentum may be added.)

Finally, given the importance of H in determining the nonlinear evolution of a system of interacting modes in a nonlinear dielectric waveguide [12,18], and its interpretation as conserved wave momentum flow, we have thus verified (for these types of systems) a comment made by Nelson that [27], '' . . . wave momentum is a more important quantity in wave interactions than either momentum or pseudomomentum alone.''

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APPENDIX A: USEFUL VECTOR AND TENSOR IDENTITIES

1. Conventions

As there are several different conventions pertaining to the expression of tensor calculus in component form, e.g., Refs. [26,46], we state our conventions here to avoid reader confusion and collect the relevant identities consistent with these conventions below.

For vectors $\mathbf{a}$ and $\mathbf{b}$ and second rank tensor $\mathbf{T}$, we define

$$[\mathbf{a}\cdot\mathbf{T}]_i=a_jT_{ji}, \quad (A1a)$$

$$[\mathbf{T}\cdot\mathbf{a}]_i=T_{ij}a_j, \quad (A1b)$$

$$[\nabla\mathbf{a}]_{ij}=\frac{\partial a_i}{\partial x_j}\equiv\partial_j a_i\equiv a_{i,j}, \quad (A1c)$$

$$[\nabla\cdot\mathbf{T}]_i=\frac{\partial T_{ij}}{\partial x_j}\equiv\partial_j T_{ij}\equiv T_{ij,j}, \tag{A1d}$$

and thus

$$[\nabla\cdot(\mathbf{ab})]_i=\partial_j(a_i b_j)\equiv(a_i b_j)_{,j}, \tag{A1e}$$

where $[*]_i$ denotes the ith component of the argument, $\partial_j \equiv \partial/\partial x_j$, $x_1\equiv x, x_2\equiv y$, and $x_3\equiv z$, and the Einstein summation convention for repeated indices is used. Note that using these conventions

$$[(\mathbf{a}\cdot\nabla)\mathbf{b}]_i=(a_j\partial_j)b_i\equiv a_j b_{i,j} \tag{A2a}$$

while

$$[\mathbf{a}\cdot(\nabla\mathbf{b})]_i=a_j(\partial_i b_j)\equiv a_j b_{j,i} \tag{A2b}$$

and so are not identical.

2. Identities

Using the conventions and notations defined above, it can be shown that

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b}), \tag{A3a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(\nabla\mathbf{b})+\mathbf{b}\cdot(\nabla\mathbf{a}), \tag{A3b}$$

$$(\nabla\times\mathbf{a})\times\mathbf{b}=(\mathbf{b}\cdot\nabla)\mathbf{a}-\mathbf{b}\cdot(\nabla\mathbf{a}), \tag{A3c}$$

$$\nabla\cdot(\mathbf{Ia}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\nabla\times\mathbf{b})\times\mathbf{a}-(\nabla\times\mathbf{a})\times\mathbf{b}, \tag{A3d}$$

$$\nabla\cdot(\mathbf{ab})=(\mathbf{b}\cdot\nabla)\mathbf{a}+(\nabla\cdot\mathbf{b})\mathbf{a}, \tag{A3e}$$

where $\mathbf{I}$ is the rank two unit tensor, Eq. (A3d) follows from Eqs. (A3b) and (A3c), and we note that $\nabla(\mathbf{a}\cdot\mathbf{b})\equiv\nabla\cdot(\mathbf{Ia}\cdot\mathbf{b})$.

Now, we want to convert Eq. (5) from the main text into something that looks similar to Eq. (4). The answer in free space is well known and (relatively) uncontentious, so we can use the fact that the electromagnetic energy-momentum tensor in matter must reduce to the electromagnetic energy-momentum tensor in free space to guide our analysis. This ansatz means that we are looking for a tensor of the form $\mathbf{ab}-(1/2)\mathbf{Ia}\cdot\mathbf{b}$, which must come from terms of the form $(\nabla\times\mathbf{a})\times\mathbf{b}$. Manipulating Eqs. (A3c)–(A3e), it can be shown that

$$\nabla\cdot(\mathbf{ab})-\frac{1}{2}\nabla\cdot(\mathbf{Ia}\cdot\mathbf{b})=(\nabla\times\mathbf{a})\times\mathbf{b}+(\nabla\cdot\mathbf{b})\mathbf{a}+\frac{1}{2}\mathbf{b}\cdot(\nabla\mathbf{a})-\frac{1}{2}\mathbf{a}\cdot(\nabla\mathbf{b}) \tag{A4}$$

from which it follows that

$$(\nabla\times\mathbf{a})\times\mathbf{b}\equiv\nabla\cdot(\mathbf{ab})-\frac{1}{2}\nabla\cdot(\mathbf{Ia}\cdot\mathbf{b})-(\nabla\cdot\mathbf{b})\mathbf{a}+\frac{1}{2}\mathbf{a}\cdot(\nabla\mathbf{b})-\frac{1}{2}\mathbf{b}\cdot(\nabla\mathbf{a}). \tag{A5}$$

APPENDIX B: FORMULA FOR $\mathbf{D}^L$ IN A DISPERSIVE MEDIUM

In accordance with usual practice, we assume the following *linear* constitutive relation between the Fourier components of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{D}}^L$:

$$\boldsymbol{\mathcal{D}}^L_\omega=\epsilon^L(\omega)\boldsymbol{\mathcal{E}}_\omega \tag{B1a}$$

$$\simeq\left(\epsilon^L(\omega_0)+(\omega-\omega_0)\frac{\partial\epsilon^L(\omega_0)}{\partial\omega}\right)\boldsymbol{\mathcal{E}}_\omega, \tag{B1b}$$

where in going from Eq. (B1a) to (B1b), we have made a first order Taylor series expansion of $\epsilon^L(\omega)$ about $\omega=\omega_0$. Thus

$$\mathbf{D}^L(\mathbf{r},t)=\frac{1}{2\pi}\int_0^\infty d\omega\,\boldsymbol{\mathcal{D}}^L_\omega e^{-i(\omega-\omega_0)t}\simeq\epsilon^L(\omega_0)\mathbf{E}(\mathbf{r},t)+i\frac{\partial\epsilon^L(\omega_0)}{\partial\omega}\frac{\partial\mathbf{E}}{\partial t}. \tag{B2}$$

APPENDIX C: CUBIC NONLINEARITY

For an intensity dependent nonlinear systems at a single frequency, the total (real) third-order polarization is given by

$$\boldsymbol{\mathcal{P}}^{\rm NL}\equiv\boldsymbol{\mathcal{P}}^{(3)}=\epsilon_0\boldsymbol{\chi}^{(3)}\vdots\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}} \tag{C1}$$

from which it follows for nonresonant (lossless) electronic responses for single frequency interactions, that [47]

$$\mathbf{P}^{\rm NL}=\epsilon_0\chi^{(3)}(2\mathbf{E}\cdot\mathbf{E}^*\mathbf{E}+\mathbf{E}\cdot\mathbf{E}\mathbf{E}^*), \tag{C2}$$

where $\chi^{(3)}=\chi^{(3)}_{xxxx}$. Thus, since

$$\left\langle\boldsymbol{\mathcal{P}}^{\rm NL}\cdot\frac{\partial\boldsymbol{\mathcal{E}}}{\partial z}\right\rangle=\mathbf{P}^{\rm NL}\cdot\frac{\partial\mathbf{E}^*}{\partial z}+\mathbf{P}^{\rm NL*}\cdot\frac{\partial\mathbf{E}}{\partial z}, \tag{C3}$$

it follows upon substituting Eq. (C2) for $\mathbf{P}^{\rm NL}$, that

$$\left\langle\boldsymbol{\mathcal{P}}^{\rm NL}\cdot\frac{\partial\boldsymbol{\mathcal{E}}}{\partial z}\right\rangle=\frac{1}{4}\frac{\partial}{\partial z}(\mathbf{P}^{\rm NL*}\cdot\mathbf{E}+\mathbf{P}^{\rm NL}\cdot\mathbf{E}^*)\equiv\frac{\partial}{\partial z}\left\langle\frac{1}{4}\boldsymbol{\mathcal{P}}^{\rm NL}\cdot\boldsymbol{\mathcal{E}}\right\rangle. \tag{C4}$$

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